

On the monotone properties of general affine surfaces under the Steiner symmetrization *

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Abstract

In this paper, we prove that, if functions (concave) ϕ and (convex) ψ satisfy certain conditions, the L_ϕ affine surface area is monotone increasing, while the L_ψ affine surface area is monotone decreasing under the Steiner symmetrization. Consequently, we can prove related affine isoperimetric inequalities, under certain conditions on ϕ and ψ , without assuming that the convex body involved has centroid (or the Santaló point) at the origin.

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1 Introduction

Affine invariants are very important tools for geometric analysis. Many powerful affine invariants arise in the extension of the Brunn-Minkowski theory – started by Lutwak, for instance, the L_p affine surface areas introduced by Lutwak in the ground breaking paper [20]. Notice that the study of the classical affine surface area even went back to Blaschke [3] in 1923. The L_p affine surface areas have been proved to be key ingredients in many applications, for instance, in the theory of valuations (see e.g. [1, 2, 13, 17, 18]), approximation of convex bodies by polytopes (see e.g., [8, 19, 29]), and information theory (for convex bodies, see e.g., [12, 25, 30, 31]). A beautiful result by Reisner, Schütt and Werner [26] even implies that the Mahler volume product (related to the famous unsolved Mahler conjecture) attains the minimum only at those convex bodies with L_p affine surface areas equal to zero for all $p \in (0, \infty)$.

*Keywords: affine surface area, L_p Brunn-Minkowski theory, affine isoperimetric inequality, Steiner symmetrization, the Orlicz-Brunn-Minkowski theory.

Recently, much effort has been made to extend the L_p Brunn-Minkowski theory to its next step: the Orlicz-Brunn-Minkowski theory, which is of great demand, (see e.g., [4, 10, 15, 16, 18, 22, 23, 33, 35]). As mentioned in [23], “this need is not only motivated by compelling geometric considerations (such as those presented in Ludwig and Reitzner [18]), but also by the desire to obtain Sobolev bounds (see [9]) of a far more general nature.” In particular, the L_p affine surface areas were extended in [16, 18]. As an example, here we give the definition for the L_ϕ affine surface area. Let $Conc(0, \infty)$ be the set of functions $\phi : (0, \infty) \rightarrow (0, \infty)$ such that either ϕ is a nonzero constant function, or ϕ is concave with $\lim_{t \rightarrow 0} \phi(t) = 0$ and $\lim_{t \rightarrow \infty} \phi(t)/t = 0$ (in this case, set $\phi(0) = 0$). Note that the function ϕ is monotone increasing. For $\phi \in Conc(0, \infty)$, the L_ϕ affine surface area of K [16, 18] takes the following form

$$as_\phi(K) = \int_{\partial K} \phi \left(\frac{\kappa_K(y)}{\langle y, N_K(y) \rangle^{n+1}} \right) \langle y, N_K(y) \rangle d\mu_K(y).$$

Here, $N_K(y)$ is an outer unit normal vector at y to ∂K —the boundary of K , $\kappa_K(y)$ is the Gaussian curvature at $y \in \partial K$, and μ_K denotes the usual surface area measure on ∂K . The standard inner product on \mathbb{R}^n is $\langle \cdot, \cdot \rangle$ and it induces the Euclidian norm $\| \cdot \|$. The L_p affine surface area for $p \geq 0$ is corresponding to $\phi(t) = t^{\frac{p}{n+p}}$. Here, for all $p \neq -n$, the L_p affine surface area of K is defined as (see e.g. [3, 20, 29])

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x),$$

and

$$as_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^n} d\mu_K(x),$$

provided the above integrals exist. A fundamental result on the L_ϕ affine surface area is the characterization theory of upper-semicontinuous $SL(n)$ invariant valuation [18]. Namely, *every upper-semicontinuous, $SL(n)$ invariant valuation vanishing on polytopes can be represented as an L_ϕ affine surface area for some $\phi \in Conc(0, \infty)$.*

(Affine) isoperimetric inequalities, such as, the celebrated Blaschke-Santaló inequality, are of particular importance to many problems in geometry. An affine isoperimetric inequality relates two functionals associated with convex bodies (or more general sets) where the ratio of the two functionals is invariant under non-degenerate linear transformations. Affine isoperimetric inequalities are arguably more useful than their better known Euclidean relatives. For instance, the classical

affine isoperimetric inequality (see [28]) gives an upper bound for the classical affine surface area in terms of volume. This inequality has very important applications in many other problems (e.g. [7, 21]). The affine isoperimetric inequalities for L_ϕ affine surface areas were established in [16]. Namely, *among all convex bodies with fixed volume and with centroids at the origin, the L_ϕ affine surface area attains its maximum at ellipsoids*. For the case of L_p affine surface areas, such inequalities were already appear in [20, 32]. We refer readers to Section 2 and [16, 18] for other general affine surfaces and their properties.

Note that these nice affine isoperimetric inequalities were established by using Blaschke-Santaló inequality, which requires the centroid (or the Santaló point) to be the origin. Removing such a restriction is one of the main motivations of this paper. Our main tool is the famous Steiner symmetrization, a powerful tool in convex geometry. It is of particular importance in proving many geometric inequalities (especially with either maximizer or minimizer to be Euclidean balls and/or ellipsoids). To fulfill the goals, one needs (1) to prove the monotone properties of objects of interest under the Steiner symmetrization, and (2) to employ the fact that each convex body will eventually approach to (in Hausdorff metric) an origin-symmetric Euclidean ball by the successive Steiner symmetrization.

In this paper, we will study monotone properties of general affine surface areas under the Steiner symmetrization, and then provide a different proof for related affine isoperimetric inequalities proved in the remarkable paper by Ludwig [16]. In literature, some special cases have been studied. For instance, in [11], Hug proved that the classical affine surface area (with respect to $\phi(t) = t^{\frac{1}{n+1}}$) is monotone increasing under the Steiner symmetrization, and hence the classical affine isoperimetric inequality follows. When K is smooth enough, the $L_{\pm\infty}$ affine surface area of K (with respect to $\phi(t) = t$) is equal to the volume of K° , the polar body of K . A well-known result in [24] states that, if K has centroid (or the Santaló point) at the origin, then the volume of K° is smaller than or equal to the volume of the polar body of the Steiner symmetral of K under any direction; hence the celebrated Blaschke-Santaló inequality follows. Readers are referred to, e.g., [11, 24] (and references therein) for related work for $p = 1, \pm\infty$.

We summarize our main results on the L_ϕ affine surface areas as follows. For all $\xi \in S^{n-1}$, the Steiner symmetral of K with respect to ξ is denoted by $S_\xi(K)$. We also use $|K|$ to denote the volume of K .

Theorem A. *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior, and $\phi \in \text{Conc}(0, \infty)$. Assume that the function $F(t) = \phi(t^{n+1})$ for $t \in (0, \infty)$ is concave.*

(i). The L_ϕ affine surface area is monotone increasing under the Steiner symmetrization, i.e., $as_\phi(K) \leq as_\phi(S_\xi(K))$ for all $\xi \in S^{n-1}$.

(ii). The affine isoperimetric inequality for the L_ϕ affine surface area holds true. That is,

$$as_\phi(K) \leq as_\phi(B_K),$$

where B_K is the origin-symmetric ball such that $|K| = |B_K|$.

Moreover, if $F(\cdot)$ is strictly concave and K has positive Gaussian curvature almost everywhere, then equality holds if and only if K is an origin-symmetric ellipsoid.

This paper is organized as follows. Section 2 is for notations and background on convex geometry, especially for general affine surface areas. The main results will be proved in Section 3. General references for convex geometry are [6, 28].

2 Background and Notations

The setting will be the n -dimensional Euclidean space \mathbb{R}^n . The standard orthonormal basis of \mathbb{R}^n are $\{e_1, \dots, e_n\}$. When we write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, we assume that e_n is associated with the last factor, and hence,

$$e_n^\perp = \{z \in \mathbb{R}^n : \langle z, e_n \rangle = 0\} = \{z \in \mathbb{R}^n : z_n = 0\}. \quad (2.1)$$

A set $L \subset e_n^\perp$ will be identified as a subset (and still denoted by L) of \mathbb{R}^{n-1} (by deleting the last coordinate, which is 0 always). A *convex body* $K \subset \mathbb{R}^n$ is a compact convex subset of \mathbb{R}^n with nonempty interior. In this paper, we always assume that the origin is in the interior of K . We use B_2^n and S^{n-1} to denote the unit Euclidean ball and sphere in \mathbb{R}^n respectively. The *polar body* of K , denoted by K° , is defined as $K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall x \in K\}$. The *support function* h_K of K is defined as $h_K(u) = \max_{x \in K} \langle x, u \rangle$, for all $u \in S^{n-1}$. The *Hausdorff distance* between two convex bodies $K, L \subset \mathbb{R}^n$, denoted by $d_H(K, L)$, is defined as

$$d_H(K, L) = \|h_K(u) - h_L(u)\|_\infty = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

A convex body K is said to have *curvature function* $f_K(u) : S^{n-1} \rightarrow \mathbb{R}$ if

$$V(L, \underbrace{K, \dots, K}_{n-1}) = \frac{1}{n} \int_{S^{n-1}} h_L(u) f_K(u) d\sigma(u),$$

where $V(L, K, \dots, K)$ is the mixed volume and σ is the classical spherical measure on S^{n-1} . The L_p curvature function for K is defined as $f_p(K, u) = h_K(u)^{1-p} f_K(u)$ (see [20]).

For a linear map or a matrix T , we use $\det(T)$ for the determinant of T . For a (smooth enough) function $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, $\nabla f(x)$ denotes its gradient function, and $\langle f(x) \rangle = f(x) - \langle x, \nabla f(x) \rangle$. Note that $\langle f(x) \rangle$ is linear; namely, for any two (smooth enough) functions $f, g : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ and for all $a, b \in \mathbb{R}$, one has $\langle af(x) + bg(x) \rangle = a\langle f(x) \rangle + b\langle g(x) \rangle$. In particular, we will often use the following special case

$$\left\langle \frac{f(x) + g(x)}{2} \right\rangle = \frac{\langle f(x) \rangle + \langle g(x) \rangle}{2}.$$

In [16], Ludwig also introduced the L_ψ affine surface area. Hereafter, $Conv(0, \infty)$ denotes the set of functions $\psi : (0, \infty) \rightarrow (0, \infty)$ such that either ψ is a nonzero constant function, or ψ is convex with $\lim_{t \rightarrow 0} \psi(t) = \infty$ and $\lim_{t \rightarrow \infty} \psi(t) = 0$ (in this case, we set $\psi(0) = \infty$). For $\psi \in Conv(0, \infty)$, the L_ψ affine surface area of K can be formulated as

$$as_\psi(K) = \int_{\partial K} \psi \left(\frac{\kappa_K(y)}{\langle y, N_K(y) \rangle^{n+1}} \right) \langle y, N_K(y) \rangle d\mu_K(y).$$

In particular, the L_p affine surface area for $-n < p < 0$ is corresponding to $\psi(t) = t^{\frac{p}{n+p}}$. Moreover, for all $\psi \in Conv(0, \infty)$, the L_ψ affine surface area is a lower-semicontinuous, $SL(n)$ -invariant valuation. Affine isoperimetric inequalities for L_ψ affine surface areas were also established in [16]. Namely, *among all convex bodies with fixed volume and with centroids at the origin, the L_ψ affine surface area attains its minimum at ellipsoids.*

The L_ϕ affine surface area for $\phi \in Conc(0, \infty)$ is proved to be upper-semicontinuous [18]. That is, for any sequence of convex bodies K_i converging to K in the Hausdorff metric $d_H(\cdot, \cdot)$, one has

$$\limsup_{j \rightarrow \infty} as_\phi(K_j) \leq as_\phi(K).$$

The L_ψ affine surface area for $\psi \in Conv(0, \infty)$ is showed to be lower-semicontinuous [16]; namely for any sequence of convex bodies K_i converging to K in the Hausdorff metric $d_H(\cdot, \cdot)$, one has

$$\liminf_{j \rightarrow \infty} as_\psi(K_j) \geq as_\psi(K).$$

The semicontinuous properties will be crucial in proving related affine isoperimetric inequalities in Section 3.

For a convex body $K \subset \mathbb{R}^n$ and on the direction e_n , let $K_{e_n} \subset e_n^\perp$ be the orthogonal projection of K onto e_n^\perp and $K_0 \subset e_n^\perp$ be the relative interior of K_{e_n} . We denote $f, g : K_{e_n} \rightarrow \mathbb{R}$ the *overgraph* and *undergraph* functions of K with respect to e_n , i.e.,

$$f(x) = \max\{t \in \mathbb{R} : (x, t) \in K\}, \quad g(x) = -\min\{t \in \mathbb{R} : (x, t) \in K\}, \quad x \in K_{e_n}.$$

Denote $K^+ := \text{graph}(f(x))$ and $K^- := \text{graph}(-g(x))$. Similarly, one can define overgraph and undergraph functions for K with respect to any direction $u \in S^{n-1}$.

Lemma 2.1 *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. For all $\phi \in \text{Conc}(0, \infty)$, one has*

$$as_\phi(K) = \int_{K_0} \left\{ \phi \left(\frac{|\det(d^2 f(x))|}{\langle f(x) \rangle^{n+1}} \right) \langle f(x) \rangle + \phi \left(\frac{|\det(d^2 g(x))|}{\langle g(x) \rangle^{n+1}} \right) \langle g(x) \rangle \right\} dx.$$

Proof. For almost all $x \in K_0$ (with respect to the $(n-1)$ -dimensional Lebesgue measure on K_0), the Gaussian curvature of the point $y = (x, f(x)) \in \partial K$ can be formulated as [14]

$$\kappa_K(y) = \frac{|\det(d^2 f(x))|}{(\sqrt{1 + \|\nabla f(x)\|^2})^{n+1}}. \quad (2.2)$$

On the other hand, the outer unit normal vector to ∂K at the point y is

$$N_K(y) = \frac{(-\nabla f(x), 1)}{\sqrt{1 + \|\nabla f(x)\|^2}}. \quad (2.3)$$

Hence, we have

$$\langle y, N_K(y) \rangle = \frac{f(x) - \langle x, \nabla f(x) \rangle}{\sqrt{1 + \|\nabla f(x)\|^2}} = \frac{\langle f(x) \rangle}{\sqrt{1 + \|\nabla f(x)\|^2}}. \quad (2.4)$$

For almost all $x \in K_0$, by formulas (2.2), (2.3), and (2.4), one has, at the point $y = (x, f(x)) \in \partial K$,

$$\frac{\kappa_K(y)}{\langle y, N_K(y) \rangle^{n+1}} = \frac{|\det(d^2 f(x))|}{\langle f(x) \rangle^{n+1}}. \quad (2.5)$$

The surface area can be rewritten as

$$d\mu_K(y) = \sqrt{1 + \|\nabla f(x)\|^2} dx.$$

Combining with the Federer's area formula (see [5]) and $f(x)$ being locally Lipschitz, one has

$$\int_{K^+} \phi \left(\frac{\kappa_K(y)}{\langle y, N_K(y) \rangle^{n+1}} \right) \langle y, N_K(y) \rangle d\mu_K(y) = \int_{K_0} \phi \left(\frac{|\det(d^2 f(x))|}{\langle f(x) \rangle^{n+1}} \right) \langle f(x) \rangle dx.$$

Similarly, $g(x)$ being locally Lipschitz and the Federer's area formula imply that

$$\int_{K^-} \phi \left(\frac{\kappa_K(z)}{\langle z, N_K(z) \rangle^{n+1}} \right) \langle z, N_K(z) \rangle d\mu_K(z) = \int_{K_0} \phi \left(\frac{|\det(d^2 g(x))|}{\langle g(x) \rangle^{n+1}} \right) \langle g(x) \rangle dx.$$

Finally, let $K' = \partial K \cap (\text{relbd}(K_0), \mathbb{R})$ where

$$(\text{relbd}(K_0), \mathbb{R}) = \{(x, t) : x \in \text{relbd}(K_0), t \in \mathbb{R}\}.$$

Then, the boundary of K can be decomposed as $K^+ \cup K^- \cup K'$. From generalized cylindrical coordinates, one gets, for $\phi \in \text{Conc}(0, \infty)$,

$$\int_{K'} \phi \left(\frac{\kappa_K(y)}{\langle y, N_K(y) \rangle^{n+1}} \right) \langle y, N_K(y) \rangle d\mu_K(y) = 0.$$

Therefore,

$$\begin{aligned} as_\phi(K) &= \int_{K^+ \cup K^-} \phi \left(\frac{\kappa_K(y)}{\langle y, N_K(y) \rangle^{n+1}} \right) \langle y, N_K(y) \rangle d\mu_K(y) \\ &= \int_{K_0} \left\{ \phi \left(\frac{|\det(d^2 f(x))|}{\langle f(x) \rangle^{n+1}} \right) \langle f(x) \rangle + \phi \left(\frac{|\det(d^2 g(x))|}{\langle g(x) \rangle^{n+1}} \right) \langle g(x) \rangle \right\} dx. \end{aligned}$$

To have similar results for the L_ψ affine surface area, one needs to assume that K has positive Gaussian curvature almost everywhere (with respect to the measure μ_K). Along the same line, one can prove the following lemma.

Lemma 2.2 *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior and having positive Gaussian curvature almost everywhere. For all $\psi \in \text{Conv}(0, \infty)$, one has*

$$as_\psi(K) = \int_{K_0} \left\{ \psi \left(\frac{|\det(d^2 f(x))|}{\langle f(x) \rangle^{n+1}} \right) \langle f(x) \rangle + \psi \left(\frac{|\det(d^2 g(x))|}{\langle g(x) \rangle^{n+1}} \right) \langle g(x) \rangle \right\} dx.$$

3 Main Results

The Steiner symmetral of K with respect to e_n , denoted by $S_{e_n}(K)$, is defined as

$$S_{e_n}(K) = \left\{ (x, t) : -\frac{1}{2}(f(x) + g(x)) \leq t \leq \frac{1}{2}(f(x) + g(x)), x \in K_{e_n} \right\}, \quad (3.6)$$

where K_{e_n} is the orthogonal projection of K onto e_n^\perp . The Steiner symmetral of K with respect to any direction $\xi \in S^{n-1}$ is denoted by $S_\xi(K)$ and can be formulated similar to formula (3.6). Clearly, the Steiner symmetrization does not change the volume; namely, $|S_\xi(K)| = |K|$ for all $\xi \in S^{n-1}$. In the later proof, we focus on the direction $e_n = (0, \dots, 0, 1)$ only.

The following well-known lemma is of particular importance in applications (see e.g. [6]).

Lemma 3.1 *Let K be a convex body in \mathbb{R}^n . There is a sequence of directions $\xi_i \in S^{n-1}$, $i \in \mathbb{N}$, such that the successive Steiner symmetral of K*

$$K_m = S_{\xi_m}(S_{\xi_{m-1}}(\dots(S_{\xi_1}(K))\dots))$$

converges to an origin-symmetric Euclidean ball in Hausdorff distance $d_H(\cdot, \cdot)$.

The following lemma is an easy consequence of Theorem G in [27] (see p. 205).

Lemma 3.2 *Let A and B be two $(n-1) \times (n-1)$ symmetric, positive semi-definite matrices. Then*

$$2 \left[\det \left(\frac{A+B}{2} \right) \right]^{\frac{1}{n+1}} \geq (\det(A))^{\frac{1}{n+1}} + (\det(B))^{\frac{1}{n+1}}.$$

If, in addition, B (or A) is positive definite, equality holds if and only if $A = B$.

In order to settle the equality conditions for general affine isoperimetric inequalities, we need the following lemma (see [11]), which follows from Brunn's classical characterization of ellipsoids. For a convex body K with the origin in its interior, and $\xi \in S^{n-1}$, we define by $M(K, \xi)$ the set of the midpoints of all line segments $K \cap L$ where L varies over all lines with direction ξ that meet the interior of K .

Lemma 3.3 *Let K be a convex body with the origin in its interior. Let S^* be a dense subset of S^{n-1} . Then K is an ellipsoid if and only if for each $\xi \in S^*$, the set $M(K, \xi)$ is contained in a hyperplane.*

The following lemma was proved in [11].

Lemma 3.4 *Let $U \subset \mathbb{R}^{n-1}$, $0 \in U$, be open and convex. Let $f : U \rightarrow \mathbb{R}$ be locally Lipschitz and differentiable at 0. If, $\langle x, \nabla f(x) \rangle = f(x)$ for almost all $x \in U$ such that f is differentiable at x , then $f(x) = \langle v, x \rangle$ for all $x \in U$ and some suitable $v \in \mathbb{R}^{n-1}$.*

3.1 L_ϕ affine surfaces are increasing under the Steiner symmetrization

Theorem 3.1 *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior and $\phi \in \text{Conc}(0, \infty)$. Suppose that the function $F(t) = \phi(t^{n+1})$ for $t \in (0, \infty)$ is concave. One has, for all $\xi \in S^{n-1}$,*

$$as_\phi(K) \leq as_\phi(S_\xi(K)).$$

Remark. Note that the function $\phi \in \text{Conc}(0, \infty)$ is monotone increasing, hence F is also an increasing function. In fact, in view of Lemma 3.2, the condition that F is concave and monotone increasing is more natural for proving Theorem 3.1. If (non-constant) function F is monotone increasing and concave with $F(0) = 0$, then $\phi \in \text{Conc}(0, \infty)$. (It is easily checked that F constant implies ϕ constant). To this end, for all $0 < t < s$, $\phi(t) = F(t^{\frac{1}{n+1}}) \leq F(s^{\frac{1}{n+1}}) = \phi(s)$ and hence ϕ is monotone increasing. Note that the function $t^{\frac{1}{n+1}}$ is concave, and by F being increasing, for all $\lambda \in [0, 1]$ and $0 < t < s$,

$$\begin{aligned} \phi(\lambda t + (1 - \lambda)s) &= F([\lambda t + (1 - \lambda)s]^{\frac{1}{n+1}}) \geq F(\lambda t^{\frac{1}{n+1}} + (1 - \lambda)s^{\frac{1}{n+1}}) \\ &\geq \lambda F(t^{\frac{1}{n+1}}) + (1 - \lambda)F(s^{\frac{1}{n+1}}) = \lambda \phi(t) + (1 - \lambda)\phi(s). \end{aligned}$$

The concavity of F implies that, for all (given) $t > 1$ and $\lambda = 1/t \in [0, 1]$,

$$F(1) = F(\lambda t + (1 - \lambda) \cdot 0) \geq \lambda F(t) + (1 - \lambda)F(0) = \frac{F(t)}{t}.$$

Hence, $F(t) \leq F(1)t$ for all $t > 1$, which further implies that

$$0 \leq \lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = \lim_{t \rightarrow \infty} \frac{F(t^{\frac{1}{n+1}})}{t} \leq \lim_{t \rightarrow \infty} F(1)t^{\frac{-n}{n+1}} = 0.$$

That is, $\lim_{t \rightarrow \infty} \phi(t)/t = 0$. In words, we have conclude that $\phi \in \text{Conc}(0, \infty)$. However, $\phi \in \text{Conc}(0, \infty)$ does not in general imply F being concave and monotone

increasing. For instance, for homogeneous function $\phi(t) = t^a$, to have F concave, one needs $0 \leq a \leq \frac{1}{n+1}$.

Proof of Theorem 3.1. Let K be a convex body with the origin in its interior. Recall that, $as_\phi(K)$ is affine invariant, i.e., for all linear maps with $|\det(T)| = 1$ and for all $\phi \in Conc(0, \infty)$, one has

$$as_\phi(TK) = as_\phi(K).$$

Hence, without loss of generality, we only prove Theorem 3.1 on the direction

$$\xi = e_n = (0, \dots, 0, 1).$$

By Lemma 2.1, the L_ϕ affine surface area of K can be rewritten as

$$as_\phi(K) = \int_{K_0} \left\{ \phi \left(\frac{|\det(d^2 f(x))|}{\langle f(x) \rangle^{n+1}} \right) \langle f(x) \rangle + \phi \left(\frac{|\det(d^2 g(x))|}{\langle g(x) \rangle^{n+1}} \right) \langle g(x) \rangle \right\} dx.$$

Let $h(x) = [f(x) + g(x)]/2$. By Lemma 3.2, one gets, for almost all $x \in K_0$,

$$\begin{aligned} 2 |\det(d^2 h(x))|^{\frac{1}{n+1}} &= 2 \left| \det \left[\frac{d^2 f(x) + d^2 g(x)}{2} \right] \right|^{\frac{1}{n+1}} \\ &\geq |\det(d^2 f(x))|^{\frac{1}{n+1}} + |\det(d^2 g(x))|^{\frac{1}{n+1}}. \end{aligned} \quad (3.7)$$

Recall that $F(t) = \phi(t^{n+1})$ and

$$\langle h(x) \rangle = \frac{\langle f(x) \rangle + \langle g(x) \rangle}{2},$$

for almost all $x \in K_0$. Hence,

$$\phi \left(\frac{|\det(d^2 h(x))|}{\langle h(x) \rangle^{n+1}} \right) = F \left(\frac{|\det(d^2 h(x))|^{\frac{1}{n+1}}}{\langle h(x) \rangle} \right) = F \left(\frac{2 |\det(d^2 h(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle + \langle g(x) \rangle} \right).$$

Note that $\phi \in Conc(0, \infty)$ is an increasing function, so is $F(t)$ on $(0, \infty)$. By inequality (3.7), one has

$$\begin{aligned} \phi \left(\frac{|\det(d^2 h(x))|}{\langle h(x) \rangle^{n+1}} \right) &\geq F \left(\frac{|\det(d^2 f(x))|^{\frac{1}{n+1}} + |\det(d^2 g(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle + \langle g(x) \rangle} \right) \\ &\geq F \left(\frac{|\det(d^2 f(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle} \right) \frac{\langle f(x) \rangle}{\langle f(x) \rangle + \langle g(x) \rangle} \\ &\quad + F \left(\frac{|\det(d^2 g(x))|^{\frac{1}{n+1}}}{\langle g(x) \rangle} \right) \frac{\langle g(x) \rangle}{\langle f(x) \rangle + \langle g(x) \rangle}, \end{aligned} \quad (3.8)$$

where the last inequality in (3.8) follows from the concavity of the function $F(t)$ on $(0, \infty)$. Therefore, by (3.6) and Lemma 2.1, we have for all $\phi \in \text{Conc}(0, \infty)$,

$$\begin{aligned} as_\phi(S_{e_n}(K)) &= 2 \int_{K_0} \left\{ \phi \left(\frac{|\det(d^2 h(x))|}{\langle h(x) \rangle^{n+1}} \right) \langle h(x) \rangle \right\} dx \\ &\geq \int_{K_0} \left\{ \phi \left(\frac{|\det(d^2 f(x))|}{\langle f(x) \rangle^{n+1}} \right) \langle f(x) \rangle + \phi \left(\frac{|\det(d^2 g(x))|}{\langle g(x) \rangle^{n+1}} \right) \langle g(x) \rangle \right\} dx \\ &= as_\phi(K). \end{aligned}$$

Let K be a convex body having curvature function $f_K : S^{n-1} \rightarrow \mathbb{R}$. For $\phi \in \text{Conc}(0, \infty)$, the L_ϕ^* affine surface area of K , denoted by $as_\phi^*(K)$, can be formulated as [16]

$$as_\phi^*(K) = \int_{S^{n-1}} \phi(f_{-n}(K, u)) d\nu_K(u),$$

where $f_{-n}(K, u) = h_K(u)^{n+1} f_K(u)$ is the L_p curvature function of K (see [20]) for $p = -n$, while $d\nu_K(u) = d\sigma(u)/h_K(u)^n$ with $d\sigma(u)$ the classical spherical measure over the sphere S^{n-1} . It was proved in [16] that

$$as_\phi^*(K) = as_\phi(K^\circ), \quad (3.9)$$

for any convex body K having curvature function and with the origin in its interior. Combining with Theorem 3.1, one immediately has the following result.

Corollary 3.1 *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior and having curvature function. Let $\phi \in \text{Conc}(0, \infty)$. Assume that the function $F(t) = \phi(t^{n+1})$ for $t \in (0, \infty)$ is concave. Then, the L_ϕ^* affine surface area is monotone increasing under the Steiner symmetrization in the following sense:*

$$as_\phi^*(K) \leq as_\phi^*([S_\xi(K^\circ)]^\circ),$$

for all $\xi \in S^{n-1}$, such that, $[S_\xi(K^\circ)]^\circ$ has curvature function.

Theorem 3.2 *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior, and B_K be the origin-symmetric Euclidean ball with $|K| = |B_K|$. Then, for all $\phi \in \text{Conc}(0, \infty)$ such that the function $F(t) = \phi(t^{n+1})$ for $t \in (0, \infty)$ is concave, one has*

$$as_\phi(K) \leq as_\phi(B_K).$$

If in addition $F(\cdot)$ is strictly concave and K has positive Gaussian curvature almost everywhere, equality holds if and only if K is an origin-symmetric ellipsoid.

Remark. Clearly, if $as_\phi(K) = as_\phi(B_K)$, then K *cannot* be a convex body with Gaussian curvature equal to 0 almost everywhere (with respect to the measure μ_K) on the boundary of K , as otherwise $as_\phi(K) = 0$.

Proof. Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. Suppose that $\phi \in Conc(0, \infty)$. By Lemma 3.1, one can find a sequence of directions $\{u_i\}_{i=1}^\infty \subset \Omega$ such that K_i converges to B_K in the Hausdorff distance. Here K_i is defined as follows:

$$K_1 = S_{u_1}(K); \quad K_{i+1} = S_{u_{i+1}}(K_i), \quad \forall i = 1, 2, \dots$$

Theorem 3.1 implies that

$$as_\phi(K) \leq as_\phi(K_1) \leq \dots \leq as_\phi(K_j), \quad \forall j \in \mathbb{N}.$$

Combining with the upper-semicontinuity of $as_\phi(\cdot)$, one has,

$$as_\phi(K) \leq \limsup_{j \rightarrow \infty} as_\phi(K_j) \leq as_\phi(\lim_{j \rightarrow \infty} K_j) = as_\phi(B_K). \quad (3.10)$$

Now let us assume that F is strictly concave and $as_\phi(K) = as_\phi(B_K)$. Let K be a convex body with positive Gaussian curvature almost everywhere. We now claim that the set $M(K, e_n)$ is contained in a hyperplane. In this case, we assume that e_n is a direction such that both the overgraph and undergraph functions f, g are differentiable at 0. Note that $as_\phi(K) = as_\phi(S_{e_n}(K))$ requires equalities for (3.8). Combining with Lemma 3.2 and the strict concavity of F , one has, for almost every $x \in K_0$,

$$|\det(d^2 f(x))| = |\det(d^2 g(x))|; \quad \frac{|\det(d^2 f(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle} = \frac{|\det(d^2 g(x))|^{\frac{1}{n+1}}}{\langle g(x) \rangle} > 0. \quad (3.11)$$

Hence, for almost all $x \in K_0$,

$$\langle f(x) \rangle = f(x) - \langle x, \nabla f(x) \rangle = g(x) - \langle x, \nabla g(x) \rangle = \langle g(x) \rangle.$$

That is, $f(x) - g(x) = \langle x, \nabla(f(x) - g(x)) \rangle$ for almost all $x \in K_0$. Note that $f - g$ is locally Lipschitz. From Lemma 3.4, one obtains that $f(x) - g(x)$ is linear, and hence $M(K, e_n)$ is contained in a hyperplane.

Let Ω be the dense subset of S^{n-1} such that the corresponding overgraph and undergraph functions at the direction $u \in \Omega$ are both differentiable at 0. Here, we have used the fact that $\sigma(S^{n-1} \setminus \Omega) = 0$, because $u \in \Omega$ if and only if the radial function ρ_K of K is differentiable at $\pm u$. For any $u \in \Omega$, there is a rotation T such

that $T(u)$ is parallel to e_n . The above claim then implies that $M(TK, T(u))$ (and hence $M(K, u)$) is contained in a hyperplane. By Lemma 3.3, K is an ellipsoid. Moreover, K has to be an origin-symmetric ellipsoid. To this end, without loss of generality, we assume that K is a ball with center $y_c \neq 0$ (this follows from affine invariance of the L_ϕ surface area). By formulas (2.5) and (3.11), one gets

$$\frac{[\kappa_K(y)]^{\frac{1}{n+1}}}{\langle y, N_K(y) \rangle} = \frac{|\det(d^2 f(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle} = \frac{|\det(d^2 g(x))|^{\frac{1}{n+1}}}{\langle g(x) \rangle} = \frac{[\kappa_K(z)]^{\frac{1}{n+1}}}{\langle z, N_K(z) \rangle}, \quad \forall x \in K_{e_n},$$

with $y = (x, f(x)) \in \partial K$ and $z = (x, -g(x)) \in \partial K$. Note that the curvature of K is a constant, and hence $\kappa_K(y) = \kappa_K(z)$. This implies $\langle y, N_K(y) \rangle = \langle z, N_K(z) \rangle$, a contradiction with K being a ball with center $y_c \neq 0$.

Combining with formula (3.9), one immediately has the following isoperimetric inequality for the L_ϕ^* affine surface area.

Corollary 3.2 *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior and having curvature function. Let B_K be the origin-symmetric Euclidean ball with $|K| = |B_K|$. Then, for all $\phi \in \text{Conc}(0, \infty)$ such that the function $F(t) = \phi(t^{n+1})$ for $t \in (0, \infty)$ is concave, one has*

$$as_\phi^*(K) \leq as_\phi^*([B_K]^\circ).$$

If in addition $F(\cdot)$ is strictly concave and K° has positive Gaussian curvature almost everywhere (with respect to μ_{K°), equality holds if and only if K is an origin-symmetric ellipsoid.

As the L_p affine surface areas for $p > 0$ are special cases of L_ϕ affine surface areas, with $\phi(t) = t^{\frac{p}{n+p}}$, we get the following result.

Corollary 3.3 *Let K be a convex body with the origin in its interior, and let $p \in (0, 1)$.*

(i) The L_p affine surface area for $p \in (0, 1)$ is monotone increasing under the Steiner symmetrization. That is, for any $\xi \in S^{n-1}$, one has

$$as_p(K) \leq as_p(S_\xi(K)).$$

(ii) The L_p affine surface areas attain their maximum at the ellipsoid, among all convex bodies with fixed volume. More precisely,

$$\frac{as_p(K)}{as_p(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{n-p}{n+p}}.$$

For convex bodies with positive Gaussian almost everywhere, equality holds if and only if K is an origin-symmetric ellipsoid.

Remark. Notice that if $p = 0$, the L_p affine surface area is equal to the volume and hence will not change under the Steiner symmetrization. The case $p = 1$ corresponds to the classical affine surface area, and this has been proved, for instance, in [11]. The L_p affine isoperimetric inequalities for $p > 1$ were first established in [20] and were extended to all $p \in \mathbb{R}$ in [32]. Comparing the condition on K in Corollary 3.3 with those in [32], here one does not require the centroid of K to be at the origin. This was first noticed in [34] by Zhang.

Proof. Let $p \in (0, 1)$ and $\phi(t) = t^{\frac{p}{n+p}}$ for $t \in (0, \infty)$. Then $\phi \in \text{Conc}(0, \infty)$. Moreover, it is easily checked that

$$F(t) = \phi(t^{n+1}) = t^{\frac{np+p}{n+p}}, \quad t \in (0, \infty)$$

is strictly concave since $0 < \frac{np+p}{n+p} < 1$. That is, $\phi(t) = t^{\frac{p}{n+p}}$ verifies conditions on Theorems 3.1 and 3.2. Therefore, (i) follows immediately from Theorem 3.1.

For (ii), one first has, by Theorem 3.2, $as_p(K) \leq as_p(B_K)$. Note that, $B_K = rB_2^n$ with

$$r = \left(\frac{|K|}{|B_2^n|} \right)^{1/n}$$

and for all $\lambda > 0$,

$$as_p(\lambda K) = \lambda^{\frac{n(n-p)}{n+p}} as_p(K).$$

Then, one has

$$\frac{as_p(K)}{as_p(B_2^n)} \leq \frac{as_p(B_K)}{as_p(B_2^n)} = \left(\frac{|K|}{|B_2^n|} \right)^{\frac{n-p}{n+p}}.$$

As F is strictly concave, by Theorem 3.2, one gets that among all convex bodies with positive Gaussian curvature almost everywhere, the L_p affine surface area attains its maximum *only* at ellipsoids.

3.2 L_ψ affine surfaces are decreasing under the Steiner symmetrization

For the L_ψ affine surface areas, one has the following theorem.

Theorem 3.3 *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior and $\psi \in \text{Conv}(0, \infty)$. Then, if the function $G(t) = \psi(t^{n+1})$ for $t \in (0, \infty)$ is convex, one has, for all $\xi \in S^{n-1}$,*

$$as_\psi(K) \geq as_\psi(S_\xi(K)).$$

Remark. Note that the function $\psi \in \text{Conv}(0, \infty)$ is monotone decreasing, hence G is also a decreasing function. In fact, in view of Lemma 3.2, the condition that G is convex and monotone decreasing is more natural for proving Theorem 3.3. If (non-constant) function G is monotone decreasing and convex with $\lim_{t \rightarrow 0} G(t) = \infty$ and $\lim_{t \rightarrow \infty} G(t) = 0$, then $\psi \in \text{Conv}(0, \infty)$. (It is easily checked that G constant implies ψ constant). To this end, it is easy to see that $\lim_{t \rightarrow 0} \psi(t) = \infty$ and $\lim_{t \rightarrow \infty} \psi(t) = 0$. For all $0 < t < s$, $\psi(t) = G(t^{\frac{1}{n+1}}) \geq G(s^{\frac{1}{n+1}}) = \psi(s)$ and hence ψ is monotone decreasing. Note that the function $t^{\frac{1}{n+1}}$ is concave, and by G being decreasing, for all $\lambda \in [0, 1]$ and $0 < t < s$,

$$\begin{aligned} \psi(\lambda t + (1 - \lambda)s) &= G([\lambda t + (1 - \lambda)s]^{\frac{1}{n+1}}) \leq G(\lambda t^{\frac{1}{n+1}} + (1 - \lambda)s^{\frac{1}{n+1}}) \\ &\leq \lambda G(t^{\frac{1}{n+1}}) + (1 - \lambda)G(s^{\frac{1}{n+1}}) = \lambda \psi(t) + (1 - \lambda)\psi(s). \end{aligned}$$

For homogeneous function $\psi(t) = t^a$, to have G convex and monotone decreasing, one needs $a \leq 0$.

Proof of Theorem 3.3. The proof of Theorem 3.3 is similar to that of Theorem 3.1. Here, for completeness, we include its proof with modification emphasized.

Without loss of generality, we assume that K has positive Gaussian curvature almost everywhere. Otherwise, if $\mu_K(\{y \in \partial K : \kappa_K(y) = 0\}) > 0$, then $as_\psi(K) = \infty$, and hence the desired result follows.

As $as_\psi(K)$ is $SL(n)$ -invariant, without loss of generality, we only work on the direction $\xi = e_n = (0, \dots, 0, 1)$. Let $h(x) = [f(x) + g(x)]/2$. Note that $\psi \in \text{Conv}(0, \infty)$ is a decreasing function, so is $G(t) = \psi(t^{n+1})$ on $t \in (0, \infty)$. By

inequality (3.7), one has

$$\begin{aligned}
\psi\left(\frac{|\det(d^2h(x))|}{\langle h(x) \rangle^{n+1}}\right) &= G\left(\frac{2|\det(d^2h(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle + \langle g(x) \rangle}\right) \\
&\leq G\left(\frac{|\det(d^2f(x))|^{\frac{1}{n+1}} + |\det(d^2g(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle + \langle g(x) \rangle}\right) \\
&\leq G\left(\frac{|\det(d^2f(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle}\right) \frac{\langle f(x) \rangle}{\langle f(x) \rangle + \langle g(x) \rangle} \\
&\quad + G\left(\frac{|\det(d^2g(x))|^{\frac{1}{n+1}}}{\langle g(x) \rangle}\right) \frac{\langle g(x) \rangle}{\langle f(x) \rangle + \langle g(x) \rangle}, \quad (3.12)
\end{aligned}$$

where inequality (3.12) follows from the convexity of the function $G(t)$ on $(0, \infty)$. Therefore, by (3.6) and Lemma 2.2, we have for all $\psi \in \text{Conv}(0, \infty)$,

$$\begin{aligned}
as_\psi(S_{e_n}(K)) &= 2 \int_{K_0} \left\{ \psi\left(\frac{|\det(d^2h(x))|}{\langle h(x) \rangle^{n+1}}\right) \langle h(x) \rangle \right\} dx \\
&\leq \int_{K_0} \left\{ \psi\left(\frac{|\det(d^2f(x))|}{\langle f(x) \rangle^{n+1}}\right) \langle f(x) \rangle + \psi\left(\frac{|\det(d^2g(x))|}{\langle g(x) \rangle^{n+1}}\right) \langle g(x) \rangle \right\} dx \\
&= as_\psi(K).
\end{aligned}$$

Let K be a convex body with curvature function. Similar to the L_ϕ^* affine surface area, the L_ψ^* affine surface area for $\psi \in \text{Conv}(0, \infty)$ can be formulated as

$$as_\psi^*(K) = \int_{S^{n-1}} \psi(f_{-n}(K, u)) d\nu_K(u).$$

Notice that the L_p affine surface area for $p < -n$ is a special case for the L_ψ^* affine surface area with $\psi(t) = t^{\frac{n}{n+p}}$. It was prove that

$$as_\psi^*(K) = as_\psi(K^\circ), \quad (3.13)$$

for all convex body K having curvature function and with the origin in its interior [16]. Combining with Theorem 3.3, one immediately has the following result.

Corollary 3.4 *Let $K \subset \mathbb{R}^n$ be a convex body having curvature function and with the origin in its interior. Let $\psi \in \text{Conv}(0, \infty)$. Assume that the function $G(t) = \psi(t^{n+1})$ for $t \in (0, \infty)$ is convex. Then, the L_ψ^* affine surface area is monotone decreasing under the Steiner symmetrization in the following sense:*

$$as_\psi^*(K) \geq as_\psi^*([S_\xi(K^\circ)]^\circ),$$

for all $\xi \in S^{n-1}$ such that, $[S_\xi(K^\circ)]^\circ$ has curvature function.

Theorem 3.4 *Let K be a convex body with the origin in its interior, and B_K be the origin-symmetric ball such that $|K| = |B_K|$. Then, for all $\psi \in \text{Conv}(0, \infty)$ such that the function $G(t) = \psi(t^{n+1})$ for $t \in (0, \infty)$ is convex, one has*

$$as_\psi(K) \geq as_\psi(B_K).$$

If in addition $G(t)$ is strictly convex, equality holds if and only if K is an origin-symmetric ellipsoid.

Proof. The proof of Theorem 3.4 is almost identical to that of Theorem 3.2. Here, we only mention the main modification.

Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. Suppose that $\psi \in \text{Conv}(0, \infty)$. As in the proof of Theorem 3.2, one can find a sequence of directions $\{u_i\}_{i=1}^\infty \subset \Omega$ such that K_i converges to B_K in the Hausdorff distance. Here K_i is defined as follows:

$$K_1 = S_{u_1}(K); \quad K_{i+1} = S_{u_{i+1}}(K_i), \forall i = 1, 2, \dots$$

Theorem 3.3 implies that

$$as_\psi(K) \geq as_\psi(K_1) \geq \dots \geq as_\psi(K_j), \quad \forall j \in \mathbb{N}.$$

Combining with the lower-semicontinuity of $as_\psi(\cdot)$, one has,

$$as_\psi(K) \geq \liminf_{j \rightarrow \infty} as_\psi(K_j) \geq as_\psi(\lim_{j \rightarrow \infty} K_j) = as_\psi(B_K).$$

Now let us assume that G is strictly convex and $as_\psi(K) = as_\psi(B_K)$. Clearly, to have $as_\psi(K) = as_\psi(B_K)$, K must have positive Gaussian curvature a.e. on ∂K , as otherwise $as_\psi(K) = \infty$.

Let K be a convex body with positive Gaussian curvature almost everywhere. We now claim that the set $M(K, e_n)$ is contained in a hyperplane. In this case, we assume that e_n is a direction such that both the overgraph and undergraph functions f, g are differentiable at 0. Equation $as_\psi(K) = as_\psi(S_{e_n}(K))$ requires equalities for (3.12). By the strict convexity of G , one has, for almost every $x \in K_0$,

$$|\det(d^2 f(x))| = |\det(d^2 g(x))|; \quad \frac{|\det(d^2 f(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle} = \frac{|\det(d^2 g(x))|^{\frac{1}{n+1}}}{\langle g(x) \rangle} > 0. \quad (3.14)$$

Hence, $f(x) - g(x) = \langle x, \nabla(f(x) - g(x)) \rangle$ for almost all $x \in K_0$. Assume that both f, g are differentiable at 0. From Lemma 4.3 in [11], one obtains that $f(x) - g(x)$ is linear, and hence $M(K, e_n)$ is contained in a hyperplane.

Let Ω be the dense subset of S^{n-1} such that the corresponding overgraph and undergraph functions are both differentiable at 0. For any $u \in \Omega$, there is a rotation T such that $T(u)$ is parallel to e_n . The above claim then implies that $M(TK, T(u))$ (and hence $M(K, u)$) is contained in a hyperplane. By Lemma 3.3, K is an ellipsoid. Moreover, K has to be an origin-symmetric ellipsoid. To this end, we assume that K is a ball with center $y_c \neq 0$. By formulas (2.5) and (3.14), one gets

$$\frac{[\kappa_K(y)]^{\frac{1}{n+1}}}{\langle y, N_K(y) \rangle} = \frac{|\det(d^2 f(x))|^{\frac{1}{n+1}}}{\langle f(x) \rangle} = \frac{|\det(d^2 g(x))|^{\frac{1}{n+1}}}{\langle g(x) \rangle} = \frac{[\kappa_K(z)]^{\frac{1}{n+1}}}{\langle z, N_K(z) \rangle}, \quad \forall x \in K_{e_n},$$

with $y = (x, f(x)) \in \partial K$ and $z = (x, -g(x)) \in \partial K$. Note that the curvature of K is a constant, and hence $\kappa_K(y) = \kappa_K(z)$. This implies $\langle y, N_K(y) \rangle = \langle z, N_K(z) \rangle$, a contradiction with K being a ball with center $y_c \neq 0$.

Combining with formula (3.13) one immediately has the following isoperimetric inequality for the L_ψ^* affine surface area.

Corollary 3.5 *Let $K \subset \mathbb{R}^n$ be a convex body having curvature function and with the origin in its interior. Let B_K be the origin-symmetric Euclidean ball with $|K| = |B_K|$. For all $\psi \in \text{Conv}(0, \infty)$ such that the function $G(t) = \psi(t^{n+1})$ for $t \in (0, \infty)$ is convex, one has*

$$as_\psi^*(K) \geq as_\psi^*([B_K^\circ]^\circ).$$

If in addition $G(\cdot)$ is strictly convex, equality holds if and only if K is an origin-symmetric ellipsoid.

The L_p affine surface areas for $p \in (-n, 0)$ are special cases of the L_ψ affine surface areas with $\psi(t) = t^{\frac{p}{n+p}}$. We have the following results.

Corollary 3.6 *Let K be a convex body with the origin in its interior, and let $p \in (-n, 0)$.*

(i) The L_p affine surface area for $p \in (-n, 0)$ is monotone decreasing under the Steiner symmetrization. That is, for any $\xi \in S^{n-1}$, one has

$$as_p(K) \geq as_p(S_\xi(K)).$$

(ii) The L_p affine surface areas attain their minimum at the ellipsoid, among all convex bodies with fixed volume. More precisely,

$$\frac{as_p(K)}{as_p(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{n-p}{n+p}}.$$

Equality holds if and only if K is an origin-symmetric ellipsoid.

Remark. The L_p affine isoperimetric inequalities for $p \in (-n, 0)$ were first established in [32]. Comparing the condition on K in Corollary 3.6 with those in [32], here one does not require the centroid of K to be at the origin. This was first noticed in [34].

Proof. Let $p \in (-n, 0)$ and $\psi(t) = t^{\frac{p}{n+p}}$ for $t \in (0, \infty)$. Then $\psi \in \text{Conv}(0, \infty)$. Moreover, it is easily checked that

$$G(t) = \psi(t^{n+1}) = t^{\frac{np+p}{n+p}}, \quad t \in (0, \infty)$$

is convex since $\frac{np+p}{n+p} < 0$. That is, $\psi(t) = t^{\frac{p}{n+p}}$ verifies conditions on Theorems 3.3 and 3.4. Therefore, (i) follows immediately from Theorem 3.3.

For (ii), one first has, by Theorem 3.4, $as_p(K) \geq as_p(B_K)$. Note that, $B_K = rB_2^n$ with

$$r = \left(\frac{|K|}{|B_2^n|} \right)^{1/n}$$

and for all $\lambda > 0$,

$$as_p(\lambda K) = \lambda^{\frac{n(n-p)}{n+p}} as_p(K).$$

Then, one has

$$\frac{as_p(K)}{as_p(B_2^n)} \geq \frac{as_p(B_K)}{as_p(B_2^n)} = \left(\frac{|K|}{|B_2^n|} \right)^{\frac{n-p}{n+p}}.$$

As G is strictly convex, by Theorem 3.4, one gets that equality holds if and only if K is an origin-symmetric ellipsoid.

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